

# Macroscopic traversable wormholes with zero tidal forces inspired by noncommutative geometry

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## Abstract

This paper addresses the following issues: (1) the possible existence of macroscopic traversable wormholes, given a noncommutative-geometry background, and (2) the possibility of allowing zero tidal forces, given a known density. It is shown that whenever the energy density describes a classical wormhole, the resulting solution is incompatible with quantum field theory. If the energy density originates from noncommutative geometry, then zero tidal forces are allowed. Also attributable to the noncommutative geometry is the violation of the null energy condition. The wormhole geometry satisfies the usual requirements, including asymptotic flatness.

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## 1 Introduction

Wormholes are tunnel-like structures in spacetime that link widely separated regions of our Universe or different universes altogether. Morris and Thorne [1] proposed the following line element for the wormhole spacetime:

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

using units in which  $c = G = 1$ . In this line element,  $b = b(r)$  is called the *shape function* and  $\Phi = \Phi(r)$  is called the *redshift function*, which must be everywhere finite to avoid an event horizon. For the shape function we must have  $b(r_0) = r_0$ , where  $r = r_0$  is the radius of the *throat* of the wormhole. A key requirement is the *flare-out condition* at the throat:  $b'(r_0) < 1$ , while  $b(r) < r$  near the throat. The flare-out condition can only be satisfied by violating the null energy condition, to be discussed later. Ordinarily, this violation implies that a wormhole can only be held open by the use of “exotic matter.”

Using an orthonormal frame, the Einstein field equations  $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$  yield the following simple interpretation for the components of the stress-energy tensor:  $T_{\hat{t}\hat{t}} = \rho(r)$ ,

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the energy density,  $T_{\hat{r}\hat{r}} = p_r$ , the radial pressure, and  $T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p_t$ , the lateral pressure. Morris and Thorne then suggested the theoretical construction of a wormhole by using the following strategy: specify the functions  $b = b(r)$  and  $\Phi = \Phi(r)$  in order to produce the desired properties. While this strategy retains complete control over the geometry, it leads to a practical problem: the engineering team must be able to come up with the materials or fields that yield the required stress-energy tensor. In this paper we will therefore adopt a mixed strategy consisting of a combination of geometric and physical specifications.

The success of the mixed strategy is going to depend on an important outcome of string theory, the realization that coordinates may become noncommutative operators on a  $D$ -brane [2, 3]. Noncommutativity replaces point-like objects by smeared objects [4, 5, 6] with the aim of eliminating the divergences that normally appear in general relativity. (The noncommutative geometry results in a fundamental discretization of spacetime due to the commutator  $[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu}$ , where  $\theta^{\mu\nu}$  is an antisymmetric matrix.)

The smearing can be modeled using a Gaussian distribution of minimal length  $\sqrt{\theta}$  instead of the Dirac delta function [5, 6, 7, 8]. An equally effective way is to assume that the energy density of the static and spherically symmetric and particle-like gravitational source has the form

$$\rho(r) = \frac{M\sqrt{\theta}}{\pi^2(r^2 + \theta)^2}; \quad (2)$$

(see Refs. [9] and [10].) Here the mass  $M$  is diffused throughout the region of linear dimension  $\sqrt{\theta}$  due to the uncertainty. The noncommutative geometry is an intrinsic property of spacetime and does not depend on particular features such as curvature.

As discussed in Ref. [11], to describe the mixed strategy for wormhole construction, we must first return to the Einstein field equations  $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$ , resulting in the following forms:

$$\rho(r) = \frac{b'}{8\pi r^2}, \quad (3)$$

$$p_r(r) = \frac{1}{8\pi} \left[ -\frac{b}{r^3} + 2 \left( 1 - \frac{b}{r} \right) \frac{\Phi'}{r} \right], \quad (4)$$

$$p_t(r) = \frac{1}{8\pi} \left( 1 - \frac{b}{r} \right) \left[ \Phi'' - \frac{b'r - b}{2r(r - b)} \Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r - b)} \right]. \quad (5)$$

Since Eq. (5) can be obtained from the conservation of the stress-energy tensor  $T^{\mu\nu}_{;\nu} = 0$ , only two of Eqs. (3)-(5) are independent. As a result, these can be written in the following form:

$$b' = 8\pi\rho r^2, \quad (6)$$

and

$$\Phi' = \frac{8\pi p_r r^3 + b}{2r(r - b)}. \quad (7)$$

The discussion in Ref. [11] assumes an equation of state,  $p_r = \omega\rho$ . Moreover, if  $\rho(r)$  is known, an example of which is Eq. (2), then  $b(r)$  can be determined from Eq. (6). Unfortunately, since  $b(r_0) = r_0$ , we see from Eq. (7) that  $\Phi'$  (and hence  $\Phi$ ) are not likely to exist, thereby leading to an event horizon. (Exceptions occur for certain special forms of  $b(r)$ .) While assigning the redshift function  $\Phi$  avoids this problem, we can see from

Eqs. (4) and (5) that we have returned to the engineering problem of having to determine the components  $T_{\hat{r}\hat{r}}$  and  $T_{\hat{\theta}\hat{\theta}}$  of the stress-energy tensor.

One of the goals in this paper is to show that macroscopic traversable wormholes with zero tidal forces may exist given a noncommutative-geometry background. Purely mathematically speaking, this goal can be accomplished by simply letting  $\Phi \equiv \text{constant}$ , so that  $\Phi' \equiv 0$ , the *zero-tidal-force solution* [1]. That is the topic of the next section. The feasibility of this approach will then be discussed in Sec. 3.

## 2 The solution

Our discussion begins with Eq. (2),

$$\rho(r) = \frac{M\sqrt{\theta}}{\pi^2(r^2 + \theta)^2}, \quad (8)$$

which immediately yields the total mass-energy  $M_\theta$  of the wormhole inside a sphere of radius  $r$ :

$$M_\theta = \int_{r_0}^r \rho(r') 4\pi(r')^2 dr' = \frac{2M}{\pi} \left( \tan^{-1} \frac{r}{\sqrt{\theta}} - \frac{r\sqrt{\theta}}{r^2 + \theta} \right). \quad (9)$$

Here the very small linear dimension  $\sqrt{\theta}$  raises a question regarding the scale. From Eq. (3),

$$b(r) = \frac{8M\sqrt{\theta}}{\pi} \int_{r_0}^r \frac{(r')^2 dr'}{[(r')^2 + \theta]^2} + r_0, \quad (10)$$

ensuring that  $b(r_0) = r_0$ . Since Eq. (10) is valid for any  $r_0$ , the wormhole can be macroscopic, but, as we will see in the next section, we also prefer a moderate throat size. So we simply assume that  $r = r_0$  is just large enough to permit passage. The resulting shape function is

$$b(r) = \frac{4M\sqrt{\theta}}{\pi} \left( \frac{1}{\sqrt{\theta}} \tan^{-1} \frac{r}{\sqrt{\theta}} - \frac{r}{r^2 + \theta} - \frac{1}{\sqrt{\theta}} \tan^{-1} \frac{r_0}{\sqrt{\theta}} + \frac{r_0}{r_0^2 + \theta} \right) + r_0. \quad (11)$$

Since  $\Phi' \equiv 0$ ,  $p_r(r)$  and  $p_t(r)$  can now be obtained directly from Eqs. (4) and (5).

*Remark:* Returning to the question of length scale, it is noted by Nicolini et al. [5] that there is no need to change the Einstein tensor in the field equations since the noncommutative effects can be implemented by modifying only the stress-energy tensor, as performed above. So the length scale need not be restricted to the Planck scale.

Next, to check the flare-out condition, we need to examine

$$b'(r) = \frac{4M}{\pi} \frac{\sqrt{\theta}}{r^2 + \theta} + \frac{4M\sqrt{\theta}}{\pi} \frac{r^2 - \theta}{(r^2 + \theta)^2}. \quad (12)$$

(Observe that  $b'(r) > 0$  since  $\theta \ll r$ .) At or near the throat,  $b'(r) < 1$  as long as  $\sqrt{\theta} \ll M$ . So the flare-out condition has been met. (We also have  $b(r) < r$  near the throat.) It should be noted that  $b'(r)$  is relatively small, so that  $b(r)$  is a slowly increasing function.

Closely related to the flare-out condition is the violation of the null energy condition at or near the throat. Indeed,

$$\rho + p_r = \frac{M\sqrt{\theta}}{\pi^2(r^2 + \theta)^2} - \frac{1}{8\pi} \frac{b(r)}{r^3}, \quad (13)$$

which is negative at or near the throat because of the small  $\theta$ .

Finally,  $\lim_{r \rightarrow \infty} \frac{b(r)}{r} = 0$ , so that the wormhole spacetime is asymptotically flat. All the conditions required for the existence of a wormhole have thereby been met.

### 3 Feasibility

A question that is always of interest in wormhole physics is whether a sufficiently far advanced civilization could in principle construct a wormhole. Based on the earlier discussion, a relatively straightforward way would be to start with what appears to be ordinary matter with a known energy density  $\rho(r)$ . By simply fixing  $M$  and  $\theta$ , we could view Eq. (8) in this light. The resulting  $b(r)$  in Eq. (11) would in principle suffice for the construction of the wormhole.

Unfortunately, having to assign  $\Phi(r)$  makes the theoretical construction more problematical since we have now returned to the engineering problem mentioned earlier. On the other hand, the assumption that  $\Phi \equiv \text{constant}$  is not only physically desirable, it also produces a simple solution. The feasibility of this approach is discussed next.

#### 3.1 The zero-tidal-force assumption

Regarding the validity of the zero-tidal-force assumption, the previous section is of no help. So we need to reconsider a topic that is usually neglected, the compatibility of *classical wormhole theory* with quantum field theory, taken up in some detail in Refs. [12, 13]. The compatibility of charged and thin-shell wormholes with quantum field theory are discussed in Refs. [14, 15], respectively. In all cases, a wormhole must satisfy an extended version of the quantum inequalities, originally proposed by Ford and Roman [16], who also showed that the exotic matter must be confined to a narrow band around the throat. In other words, the highly problematical exotic region should be made as small as possible. It is shown in Refs. [12, 13] that this can only be accomplished by fine-tuning the metric coefficients. More precisely, to satisfy the extended quantum inequalities, one must strike a balance between reducing the size of the exotic region and the degree of fine-tuning of the metric coefficients required to achieve this reduction. In particular,  $\Phi'(r)$  has to be fine-tuned to remain in a narrow range. The most important conclusion for present purposes is that the value  $\Phi'(r) \equiv 0$  is outside this range, so that the resulting wormhole cannot be compatible with quantum field theory. This finding also explains why none of the wormhole solutions in Ref. [1], which assumes  $\Phi'(r) \equiv 0$  throughout, satisfy the quantum inequalities, first pointed out in Ref. [16].

### 3.2 Noncommutative geometry

The previous subsection dealt with the energy density of ordinary matter and subsequent incompatibility with quantum field theory. The situation is entirely different for  $\rho(r)$  in Eq. (2), being a consequence of noncommutative geometry: the quantum inequalities, which assume an inertial Minkowski spacetime without boundary, no longer apply. So the ability to assume zero tidal forces becomes a critical consequence of the noncommutative-geometry background. An equally important consequence is that the violation of the null energy condition can largely be attributed to this geometric background rather than to “exotic matter.” Thus, since  $\sqrt{\theta} \ll M$ ,

$$\rho(r) + p_r(r) = \frac{M\sqrt{\theta}}{\pi^2(r^2 + \theta)^2} + \frac{1}{8\pi} \left[ -\frac{b}{r^3} + 2 \left( 1 - \frac{b}{r} \right) \frac{\Phi'}{r} \right] < 0$$

near  $r = r_0$  even if  $\Phi'(r) \neq 0$ . At the very least, the condition would hold for a large class of redshift functions.

### 3.3 More fine-tuning

Based on the above results, we conclude that if noncommutative geometry is a correct model, then the laws of physics seem to allow macroscopic traversable wormholes with zero tidal forces, while classical general relativity does not.

It becomes apparent, however, that from the standpoint of physical construction by an advanced civilization, there are additional practical considerations reminiscent of the fine-tuning problems in Refs. [12, 13]: due to the smearing effect,  $b(r)$  in Eq. (11) is a very slowly increasing function of the radial coordinate  $r$ , as a result of which the total mass inside a sphere of radius  $r$  increases very slowly as well. So the construction would require considerable fine-tuning even for relatively small throat sizes. More fine-tuning would be required just to determine  $\rho(r)$ . (We may assume that a sufficiently far advanced civilization could make such a determination, even if this is well beyond our own capability.) Sufficiently close to Einstein gravity, i.e., for sufficiently small  $\theta$ , the construction becomes ever more difficult, as one would expect.

## 4 Zero tidal forces: other motivations

While our main concern regarding zero tidal forces is traversability, other motivations exist. For example, in the discussion of self-contained phantom wormholes in semi-classical gravity, Garattini and Lobo [17] assumed the equation of state  $p_r(r) = \omega(r)\rho(r)$ ,  $\omega(r) < -1$ . The variable parameter  $\omega(r)$  generalizes the cosmological equation of state  $p = \omega\rho$ ,  $\omega < -1$ , representing phantom energy. The restricted choices for  $\Phi(r)$  and  $\omega(r)$  discussed include the constant redshift function which results in a reduced form of the curvature scalar:  $R(r) = 2b'(r)/r^2$ . (See Ref. [17] for details.)

In the present paper, the case for zero tidal forces can also be strengthened if a noncommutative-geometry background is assumed in a slightly modified gravitational theory, as defined in Ref. [18]. The definition is based on the gravitational field equations

in the form used by Lobo and Oliveira [19] for  $f(R)$  modified gravity under the assumption that  $\Phi'(r) \equiv 0$ :

$$\rho(r) = F(r) \frac{b'(r)}{r^2}, \quad (14)$$

$$p_r(r) = -F(r) \frac{b(r)}{r^3} + F'(r) \frac{rb'(r) - b(r)}{2r^2} - F''(r) \left[ 1 - \frac{b(r)}{r} \right], \quad (15)$$

$$p_t(r) = -\frac{F'(r)}{r} \left[ 1 - \frac{b(r)}{r} \right] + \frac{F(r)}{2r^3} [b(r) - rb'(r)], \quad (16)$$

where  $F = \frac{df}{dR}$ . The curvature scalar  $R(r)$  is again given by

$$R(r) = \frac{2b'(r)}{r^2}. \quad (17)$$

Comparing Eqs. (14) and (17), a slight change in  $F$  results in a slight change in  $R$ , which characterizes  $f(R)$  modified gravity. As noted in Ref. [18], we may quantify the notion of slightly modified gravity by assuming that  $F(r)$  remains close to unity and relatively “flat,” i.e., both  $F'(r)$  and  $F''(r)$  remain relatively small in absolute value.

To apply these ideas to the present study, it is sufficient to examine Eq. (14):

$$b'(r) = \frac{r^2}{F(r)} \frac{M\sqrt{\theta}}{\pi^2(r^2 + \theta)^2}. \quad (18)$$

Assuming that  $F(r) > 0$  for all  $r$ , we obtain  $0 < b'(r) < 1$  since  $\sqrt{\theta} \ll M$ . So the shape function satisfies the flare-out condition. Since we are now dealing with  $f(R)$  modified gravity, the quantum inequalities do not apply, once again rescuing the zero-tidal-force assumption.

## 5 Conclusion

This paper deals with the possible existence and, in a limited way, the theoretical construction of macroscopic traversable wormholes in a noncommutative-geometry setting. We relied on a mixed strategy by assuming a known energy density  $\rho(r)$  and then assigning the redshift function  $\Phi(r)$ . (Attempting to determine  $\Phi(r)$  from the Einstein field equations often leads to an event horizon.) Once  $\Phi(r)$  is assigned, calculating the pressure leads to the engineering problems discussed in Ref. [1]: one has to determine the materials or fields that lead to the remaining components of the stress-energy tensor.

One of the goals in this paper is to obtain a macroscopic wormhole with zero tidal forces, i.e.,  $\Phi'(r) \equiv 0$ . If  $\rho(r)$  is known, then the zero-tidal-force assumption allows the determination of both  $p_r$  and  $p_t$  directly. The energy density  $\rho(r)$  may correspond to ordinary matter in classical wormhole theory, but as we saw in Sec. 2, we may also have

$$\rho(r) = \frac{M\sqrt{\theta}}{\pi^2(r^2 + \theta)^2},$$

based on the assumption that in noncommutative geometry, point-like particles are replaced by smeared objects, so that the mass  $M$  is diffused throughout the region due to the uncertainty.

The difference between the two scenarios is that in the classical case, the zero-tidal-force assumption makes the wormhole solution incompatible with quantum field theory. More precisely, this solution cannot satisfy the extended quantum inequalities. Since these are based on an inertial Minkowski spacetime without boundary, they do not apply to wormholes in noncommutative geometry. So the zero-tidal-force solution is allowed. It was also determined that, under fairly general conditions, the violation of the null energy condition is due to the noncommutative geometry, rather than to exotic matter.

As discussed in Sec. 3, for classical wormholes to be compatible with quantum field theory, one must strike a balance between reducing the size of the exotic region and the degree of fine-tuning required to achieve this reduction. While this problem is avoided in noncommutative geometry, fine-tuning would still be required in the theoretical construction of the wormhole because the shape function  $b(r)$  rises very slowly due to the smearing effect, even with relatively small throat sizes. Even more fine-tuning may be required to determine  $\rho(r)$ .

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